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How should acoustics adapt to meet future demands?

Designing bells with tuned overtones

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ABSTRACT

Modern western bells are cast with basic axial symmetry and the same general 'bell' shape. The size of the bell determines the fundamental frequency. The exact profile of the bell, however, varies between founders and has usually been designed using empirical techniques to produce a bell with a clear recognisable pitch. This generally means that the fundamental and significant partials are tuned to particular values and radiate clearly. In this paper, we take a more modern approach and use numerical methods to design bell profiles with appropriately tuned modes of vibration.

INTRODUCTION

Bells are among the oldest of musical instruments and are spread throughout many cultures. The modern western bell, which is prevalent today in churches and carillons, began its major development in the seventeenth century when European bell founders discovered how to manufacture bells that exhibited a clear and recognisable pitch. The founders designed bell profiles that resulted in the tuning of not only the fundamental but also the upper partials. (Fletcher & Rossing, 1998). The historic approach to the design and manufacture of tuned bells was very often secretive and developed by a foundry through trial and error. Analysis of the sound produced when a clapper strikes a bell has since shown that there are five important modes for a tuned bell.

Modern mathematical analysis techniques can simulate a structure and determine the natural modes of vibration. Furthermore, with the advent of faster computers, numerical techniques can be used not just for analysis, but also to design structures. In previous work (Petrolito & Legge, 2005) we have outlined the use of constrained optimisation for the general design of musical structures, providing a suitable balance between constraints and desirable attributes. Numerical techniques are not new to the design of bells and were first harnessed in attempts to design major third bells (Schoof *et al*, 1987). Nevertheless, our approach is substantially different and involves frequency requirements being taken as constraints. We have illustrated how this approach may be used for the design of xylophone bars, plates and bells (Legge and Petrolito, 2007).

We now confine ourselves to the design of a specific bell. In this paper, we take a closer look at the use of constrained optimisation to design the profile of harmonically tuned bell with a Hum note of c/2 and compare it with the profile of the ancient bell founders.

MATHEMATICAL MODEL

The goal is to design a bell that responds with specified frequencies when struck in the transverse direction. The first step is to decide on a model to describe the structure and from this, to formulate the equations that govern the motion.

A bell is a three-dimensional structure. Nevertheless, as long as its thickness is small in comparison to its height and breadth, a two-dimensional shell structure is sufficient to model it. In this paper, we have modelled the bell using triangular flat shell elements based on Mindlin's thick plate theory (Mindlin, 1951) for the transverse behaviour, and two-dimensional plane stress theory (Dym and Shames, 1973) for the in-plane behaviour. Whatever the model, exact solutions are only possible for simple geometries and constant material properties. Given that our geometry is not simple and that we will be attempting to tune the bell by varying physical parameters, an exact solution is not possible and a numerical solution is therefore sought.

Problem formulation

The design of a structure that will respond in a predetermined manner involves describing the geometry by a number of parameters and varying them to produce the required outcome. There are various options for describing the profile of the bell. We have chosen to describe it in terms of the thickness, the radius and the height. Figure1 indicates how the thickness, t, and the radius, r, vary with height, z.

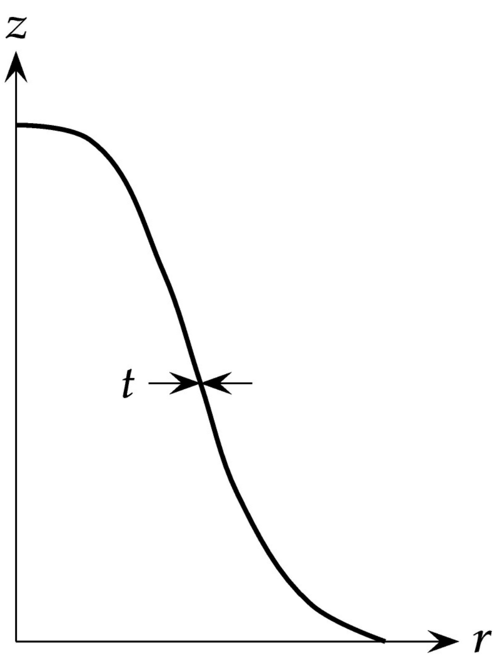

Figure 1. Bell profile described in terms of thickness, radius and height.

An optimization procedure is used to determine the values of *r* and *t* at specific heights, such that the bell has the desired frequency characteristics and satisfies any other specified criteria.

Numerical solution

We begin with the material properties and geometric parameters that roughly describe a typical c/2 bell with hum frequency approximately 523 Hz. The bell is constructed from a bell metal with density and Young's modulus ρ = 8600 kg m^3 and E = 98.6 GPa, respectively. Poisson's ratio is taken to be 0.3. The initial bell profile is described by specified co-ordinates defined by *z* and *r* as outlined in Table 1.

Table 1. Specified points on the initial bell profile

r [mm]	z [mm]	t [mm]
200.0	0.0	0.0
178.8	25.3	33.0
122.1	128.7	15.5
106.7	278.3	11.0
69.1	301.1	16.0
56.1	306.5	17.0
0.0	306.5	17.0

Each segment was assumed to be linear and further divided into equal pieces, so that the initial profile consisted of twenty-four equal spaced vertical concentric sections, and is capped with two, near horizontal pieces. This initial shape is depicted, together with its thickness, in Figure 2.

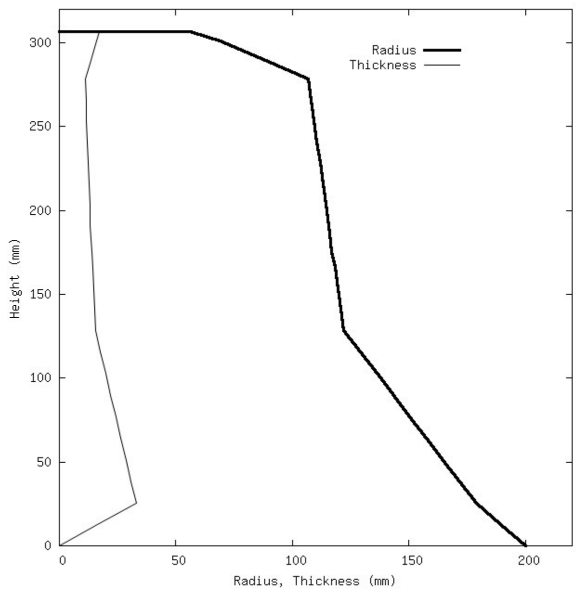

Figure 2. Initial bell profile described by thickness and radius at particular nodes.

Thus, the problem is now to determine the value of the thickness and radius for each of the nodes such that the structure will respond with the specified frequencies.

The extent of variation in radius was kept within reasonable limits so that the profile had a radius of 200 mm at the base of the bell and the radius decreased with increasing height. The thickness was set at zero at the base of the bell (lowest node) and thereon allowed to vary between 1 and 80 mm.

The accuracy of the numerical solution is dependent on a suitable mesh, with higher modes requiring greater refinement. We have chosen to use triangular flat shell elements, and these are generated by diagonally splitting square elements. Hence, to obtain accuracy to within 1% for each of the frequency constraints it was necessary to arrange the mesh so that there were 144 circumferential elements and each of the twenty-six vertical segments was further divided into three. The mesh is depicted in Figure 3.

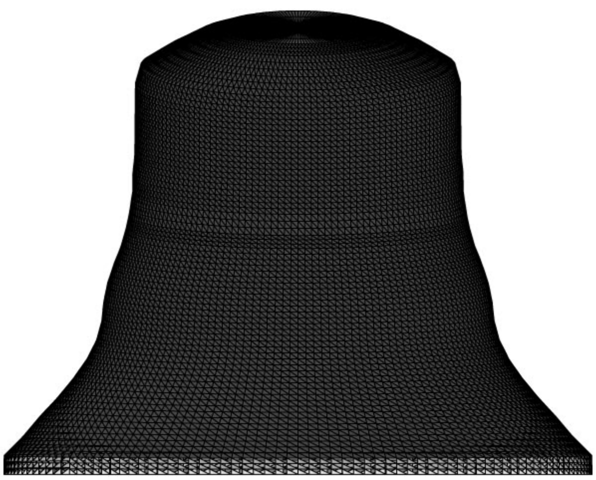

Figure 3. Mesh used to achieve suitable accuracy.

Constraints

The constraints are the desired frequencies of the five lowest modes of vibration, together with obvious structural integrity and manufacturing issues. For simplicity, we initially chose a piecewise linear variation in both the thickness and the radius. The coordinates at $z = 0.00$ and $z = 306.5$ mm were given only a small range within which they could vary. The purpose of this was to determine whether a profile, somewhat similar to the typical bell was a suitable solution. A second result was obtained by varying the radius using a piecewise-cubic polynomial approximation. This smoother profile avoids sudden jumps in the profile and subsequent complex stress systems, which the two-dimensional model may fail to accurately account for.

The modern bell has a basic axial symmetry and forms degenerate pairs for many of its normal modes of vibration. The modal patterns contain nodal meridians and circles and are labelled (m,n) accordingly. The modes that relate to the radiated sound generally have motion that is normal to the bell surface and thus do not include twisting, breathing (m=0) or swinging (m=1) modes. Hence the five lowest modes most important for a tuned bell are the first five extensional modes. (Fletcher & Rossing, 1998). These modes start with m = 2, are labelled Hum (2,0), Prime (2,1#), Teirce (3,1), Quint (3,1#) and Nominal (4,1) and are generally tuned so that their frequencies are harmonically related and in the ratio 1:2:2.4:3:4.

For our c/2 bell, the first five unique frequencies were specified as 523.3 1046.5 1244.5 1568 and 2093 Hz. The profile was required to have structural integrity (minimum allowable thickness) and normal manufacturing requirements should be met (no reverse radii, thus allowing for easy removal of the bell from its mould after casting).

RESULTS

The natural frequencies of the initial structure are given in Table 2 assuming a completely unrestrained bell and ignoring the rigid body modes. Note that, with the exception of the 9th mode, the first 11 modes are in fact degenerate pairs, so in tuning the profile we actually request modes 1, 3, 5, 7 and 10 to be assigned the specified values.

Table 2. Natural frequencies of initial bell profile

Mode no.	Natural Frequency (Hz)	Requested Frequency (Hz)
1, 2	507	523.3
3, 4	1096	1046.5
5, 6	1224	1244.5
7, 8	1829	1568
10,11	2102	2093

The results indicate that, given the crude imitation of a bell profile depicted in the initial structure, the natural frequencies are a fair representation of the harmonic series we are seeking.

The actual requested frequencies are obtained for profiles depicted in Figure 4 and Figure 5. The solutions differ in the way in which the radius is allowed to vary, and confirm that there is more than one solution to the problem.

The result in Figure 4 was obtained using a piecewise-linear variation in both thickness and radius. It serves to illustrate one possible solution for which the 5 important modes will be correctly tuned, although the abrupt angles may not conform to an ideal bell shape. Furthermore it is conceivable that stress concentrations may occur at sharp angles.

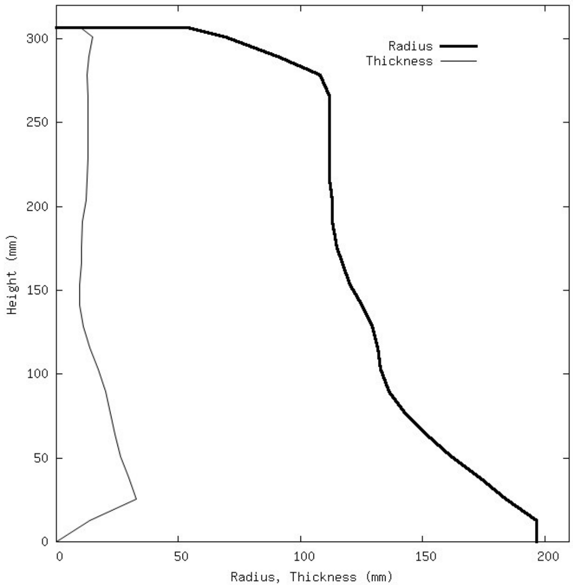

Figure 4. Bell profile for a piecewise linear variation in thickness and radius.

Figure 5 depicts a smoother profile that was obtained by varying the radius with a piecewise-cubic polynomial. It is a similar shape to the profile depicted in Figure 4 but with the sharp angles somewhat smoothed.

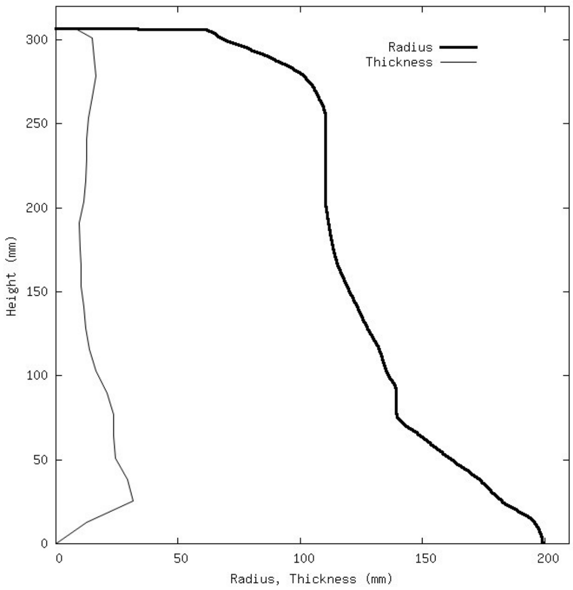

Figure 5. Bell profile for a piecewise linear variation in thickness and piecewise cubic variation in radius.

Other profiles can be generated by varying constraints, such as those placed on the radius or thickness, or by introducing an optimising function to include a particular design element.

In general, the results illustrate that constrained optimisation, where the frequency requirements are taken as constraints, is an appropriate numerical method for the design of bells. Furthermore, they confirm the seventeenth century, empirically designed ‘bell’ shape as being particularly suitable for producing bells with tuned upper partials.

CONCLUSIONS

The optimisation of the modern western bell has historically been the preserve of the bell founder, following empirical methods that attempt to optimise the tuning of the lowest 5 natural modes. In this paper, we have indicated how a mathematical approach that constrains the design to the required frequency regime can be used to design an appropriate bell profile. The technique is limited only by the adequacy of the model used to describe the structure and the ease with which constraints and optimising functions can be described mathematically.

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